Exam Geometry - June 19, 2018

Note: This exam consists of **four** problems. Usage of the theory and examples of Chapters 1:1-5, 2:1-5, 3:1-3, 4:1-6 of Do Carmo's textbook is allowed. You may not use the results of the exercises, *with the exception of* the results of Exercise 1-5:2,12, 4-3:1,2. Give a precise reference to the theory and/or exercises you use for solving the problems.
You get 10 points for free.

All functions, curves, surfaces, parametrizations and (normal) vector fields in the exam problems are differentiable, i.e., of class C^∞.

Problem 1 $(4+4+4+4+4=20$ pt.)
Let $\alpha : I \to \mathbb{R}^3$ be a unit speed curve. Here I is an interval in $\mathbb{R}$, with $0 \in I$ and $\alpha(0) = (0,0,0)$. The torsion $\tau(0)$ is positive. The binormal vector at $\alpha(s)$ is given by

$$\mathbf{b}(s) = (a\cos\varphi(s), a\sin\varphi(s), b)^T,$$

where $\varphi : I \to \mathbb{R}$ is a function with $\varphi'(s) > 0$ for $s \in I$, and a and b are positive constants with $a^2 + b^2 = 1$. Determine, for $s \in I$:

1. The torsion $\tau(s)$.
2. The normal $\mathbf{n}(s)$.
3. The tangent vector $\mathbf{t}(s)$.
4. The curvature $k(s)$.
5. The point $\alpha(s)$. (Your expression for $\alpha(s)$ may involve integrals.)

Problem 2 (5+5+5+5=20 pt.)
In each of the following cases, give an example of a surface S and two distinct points p and q on S with the required property:

1. There is exactly one geodesic of S through p and q.
2. There is no geodesic of S through p and q.
3. There are uncountably many geodesics of S through p and q.
4. The number of geodesics of S through p and q is countably infinite[1].

In each case, argue why this situation occurs.
(Note: by a geodesic we mean the subset of S which is the trace of a parametrized unit-speed geodesic. So, for example, if $\gamma(s)$ is a parametrization of a geodesic, then $\gamma(s+1)$ is a different parametrization, but of *the same* geodesic.)

Assignments 3 and 4 on next page

[1] Regarding the notions of *uncountable* and *countably infinite*, recall that, for example, there are uncountably many real numbers. The cardinality of the set of integers is countably infinite

Problem 3 (5+5+8+7=25 pt.)
The surface $S \subset \mathbb{R}^3$ has a regular parametrization $\mathbf{x} : U \to S$, with U a connected open subset of $\mathbb{R}^2$ on which the coefficients E and G of the first fundamental form are constant. Moreover, it is given that the parameter curves[2] are asymptotic curves of S. The goal is to prove that the Gaussian curvature is constant on $\mathbf{x}(U)$.

The proof consists of the following steps. Let e, f and g be the coefficients of the second fundamental form with respect to $\mathbf{x}$. The partial derivatives with respect to the parameters u and v are denoted by subscripts. For instance, $f_u = \frac{\partial f}{\partial u}, \mathbf{x}_{uv} = \frac{\partial^2 \mathbf{x}}{\partial u \partial v}$.

1. Prove that $e = g = 0$.
2. Let N be a (differentiable) unit normal field of $\mathbf{x}(U)$. Prove that $\mathbf{x}_{uv} = fN$.
3. Let Γ_{ij}^k, $i, j, k = 1, 2$, denote the Christoffel symbols with respect to $\mathbf{x}$. Then $f_u = \Gamma_{11}^1 f$ and $f_v = \Gamma_{22}^2 f$. Prove the first of these identities.
4. The Gaussian curvature of S at the point $\mathbf{x}(u, v)$ is denoted by $K(u, v)$. Then $K_u = 0$ and $K_v = 0$, for $(u, v) \in U$. Prove the first of these identities, and show that these identities together imply that K is constant on U.

Problem 4 (5+5+8+7=25 pt.)
Let $\alpha : I \to S$ be a unit-speed parametrization of a curve C on a regular surface S in $\mathbb{R}^3$. Here I is an open interval in $\mathbb{R}$. Let N be a (differentiable) unit normal field on S. The normal vector at the point $\alpha(s)$ is denoted by $N(s)$. The orthonormal frame consisting of the unit vectors $T(s) = \alpha'(s)$, $N(s)$ and $V(s) = N(s) \wedge T(s)$ is called the Darboux frame of the curve.

1. Prove that there are differentiable functions $k_n, k_g : I \to \mathbb{R}$ such that
$$\alpha''(s) = k_n(s)N(s) + k_g(s)V(s),$$
where $k_n(s)$ is the normal curvature and $k_g(s)$ is the geodesic curvature of the curve at $\alpha(s)$.
2. Suppose C is an asymptotic curve of S. Prove that C is a geodesic of S if and only if C is part of a straight line in $\mathbb{R}^3$.

Suppose the coefficients of the first fundamental form of a regular parametrization $\mathbf{x} : U \to S$ satisfy $E = G$ and $F = 0$. Take $N = \dfrac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$.
A u-curve is a curve with parametrization $u \mapsto \mathbf{x}(u, v_0)$, with v_0 constant and u ranging over an interval for which $(u, v_0) \in U$. A v-curve is defined analagously.

3. Prove that the geodesic curvature of a u-curve is given by $k_g = -\dfrac{E_v}{2E^{3/2}}$.
4. Suppose that the geodesic curvature of every u-curve is constant (along the curve, so it may be different for two distinct u-curves). Prove that every v-curve has constant geodesic curvature.

[2] These are the curves of the form $u \mapsto \mathbf{x}(u, v_0)$ and $v \mapsto \mathbf{x}(u_0, v)$, with v_0 and u_0 constant.

Solutions

Problem 1.

1. Since the curve has unit speed, we use the Frenet formulas (Chapter 1-5). In particular,

$$\mathbf{b}'(s) = (-a\varphi'(s)\sin\varphi(s), a\varphi'(s)\cos\varphi(s), 0)^T, \tag{1}$$

$\tau(s)^2 = \mathbf{b}'(s)\cdot\mathbf{b}'(s) = a^2\varphi'(s)^2$. Note that $a > 0$ and $\varphi'(s) > 0$. In particular, $\tau(s) \neq 0$, for $s \in I$. Since $\tau(0) > 0$, it follows that $\tau(s) = a\varphi'(s)$.

2. Using the Frenet formula $\mathbf{b}' = \tau\,\mathbf{n}$, and the fact that $\tau(s) = a\varphi'(s) > 0$, identity (1) yields

$$\mathbf{n}(s) = (-\sin\varphi(s), \cos\varphi(s), 0)^T.$$

3. Using $\mathbf{t}(s) = \mathbf{n}(s)\wedge\mathbf{b}(s)$ yields

$$\mathbf{t}(s) = (b\cos\varphi(s), b\sin\varphi(s), -a)^T.$$

4. Use $\mathbf{t}'(s) = k(s)\mathbf{n}(s)$ to conclude that $k(s) = b\varphi'(s)$.

5. Since $\alpha'(s) = \mathbf{t}(s)$ and $\alpha(0) = (0,0,0)$, we get

$$\alpha(s) = (b\int_{u=0}^{s}\cos\varphi(u)du, b\int_{u=0}^{s}\sin\varphi(u)du, -as).$$

Problem 2.

1. Let $S = \mathbb{R}^2$. Geodesics are straight lines. There is exactly one straight line through p and q. (Remark: one may consider S as a regular surface in $\mathbb{R}^3$ by identifying S with the xy-plane.)

2. Let $S = \mathbb{R}^2\setminus\{(0,0)\}$, and let $p = (-1,0)$ and $q = (1,0)$. Geodesics of S are straight lines not containing $(0,0)$, and open half-lines emanating from $(0,0)$ (in other words, a connected component of a set of the form $L\setminus\{(0,0)\}$, where L is a line through $(0,0)$. The points p and q determine a unique line L through $(0,0)$, but they belong to different connected components of $L\setminus\{(0,0)\}$.

3. Let S be the unit sphere in $\mathbb{R}^3$. Geodesics of S are great circles. If p and q are antipodal points of S (i.e., $q = -p$), there are uncountably many distinct geodesics (great circles) of S through p and q.

4. Let S be the circular cylinder in $\mathbb{R}^3$ with equation $x^2 + y^2 = 1$. In Example 4 in Chapter 4-4 it is shown that all geodesics on S are helices, with unit-speed parametrization $\gamma(s) = (\cos as, \sin as, bs)$ with $a^2 + b^2 = 1$. Let $p = (1,0,0)$ and let $q = (1,0,1)$. Then $\gamma(0) = p$ and $\gamma(s) = q$ iff there is an integer n such that $as = 2n\pi$ and $bs = 1$. Since $a^2 + b^2 = 1$, this shows that there are countably many distinct geodesics through p and q (and no more). More precisely,

$$a = \pm\frac{2n\pi}{\sqrt{4n^2\pi^2+1}}, \quad b = \pm\frac{1}{\sqrt{4n^2\pi^2+1}}, \quad s = \pm\sqrt{4n^2\pi^2+1}.$$

Problem 3.
1. The normal curvatures in the directions $\mathbf{x}_u$ and $\mathbf{x}_v$ are zero, so the coefficients of the second fundamental form satisfy $e = -\langle N_u, \mathbf{x}_u\rangle = 0$ and $g = -\langle N_v, \mathbf{x}_v\rangle = 0$.

2. First we prove that $\mathbf{x}_{uv}$ is parallel to N by proving that it is perpendicular to $\mathbf{x}_u$ and $\mathbf{x}_v$. This follows from $\langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle = \frac{1}{2}\langle \mathbf{x}_u, \mathbf{x}_u\rangle_v = \frac{1}{2}E_v = 0$. A similar derivation shows that $\langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle = 0$. Therefore, $\mathbf{x}_{uv} = \langle \mathbf{x}_{uv}, N\rangle\, N = fN$.

3. First observe that $f = \langle N, \mathbf{x}_{uv}\rangle$, so

$$f_u = \langle N, \mathbf{x}_{uuv}\rangle + \langle N_u, \mathbf{x}_{uv}\rangle = \langle N, \mathbf{x}_{uuv}\rangle + f\,\langle N_u, N\rangle = \langle N, \mathbf{x}_{uuv}\rangle,$$

since $\langle N_u, N\rangle = \frac{1}{2}\langle N, N\rangle_u = 0$. So we have to prove that $\langle N, \mathbf{x}_{uuv}\rangle = f\Gamma_{11}^1$. Since $\langle \mathbf{x}_{uu}, N\rangle = e = 0$, we see that

$$\mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v.$$

Hence

$$\mathbf{x}_{uuv} = (\Gamma_{11}^1)_v\, \mathbf{x}_u + (\Gamma_{11}^2)_v\, \mathbf{x}_v + \Gamma_{11}^1 \mathbf{x}_{uv} + \Gamma_{11}^2 \mathbf{x}_{vv}.$$

So

$$\langle N, \mathbf{x}_{uuv}\rangle = \Gamma_{11}^1 \langle N, \mathbf{x}_{uv}\rangle + \Gamma_{11}^2 \langle N, \mathbf{x}_{vv}\rangle = \Gamma_{11}^1 f + \Gamma_{11}^2 g = \Gamma_{11}^1 f.$$

4. Since $K = \dfrac{eg - f^2}{EG - F^2} = \dfrac{-f^2}{EG - F^2}$, and E and G are constant, we get

$$\begin{aligned} K_u &= -\frac{2ff_u}{EG - F^2} - \frac{2f^2 F F_u}{(EG - F^2)^2} \\ &= -\frac{2f^2}{EG - F^2}\big(\Gamma_{11}^1 + \frac{F\,F_u}{EG - F^2}\big) \\ &= 0. \end{aligned}$$

The last equality follows by solving Γ_{11}^1 from the first pair of equations in (2) in Chapter 4-3, using $E_u = E_v = 0$. One similarly proves that $K_v = 0$ (you don't have to do this). Therefore, K is locally constant on a connected set, so it is constant.

Remark. You may also use the Mainardi-Codazzi equation (6) in Chapter 4-3. For this you need to show that $\Gamma_{12}^2 = 0$, which follows from the second pair of equations in (2) in Chapter 4-3.

Problem 4.
1. The parametrized curve has unit speed, so $T(s) = \alpha'(s)$ and $\langle \alpha'(s), \alpha''(s)\rangle = 0$. Therefore, $\alpha'' = k_n N + k_g V$, with

$$k_n(s) = \langle \alpha''(s), N(s)\rangle = -\langle N'(s), \alpha'(s)\rangle = \mathrm{II}_{\alpha(s)}(\alpha'(s)),$$

which is the normal curvature of the curve at $\alpha(s)$. Furthermore, $k_g(s)V(s)$ is the projection of $\alpha''(s)$ onto $T_{\alpha(s)}S$, so $k_g(s)$ is the geodesic curvature of the curve at $\alpha(s)$.

2. The normal curvature of an asymptotic curve is zero (by definition), so $k_n = 0$. If C is also a geodesic, its geodesic curvature k_g is zero. In that case $\alpha''(s) = 0$ for all $s \in I$, so $\alpha(s) = \alpha(s_0) + (s - s_0)\alpha'(s_0)$, for an arbitrary, but fixed, $s_0 \in I$. Therefore, C is a straight line (since $\alpha'(s_0)$ is a unit vector, so non-zero).
Conversely, if C is a straight line, its unit-speed parametrization α satisfies $\alpha''(s) = 0$, for all $s \in I$. Therefore, $k_g = 0$, so C is a geodesic.

3. Let $\alpha(s) = \mathbf{x}(u(s), v_0)$ be a unit-speed parametrization of such a curve. Then

$$\alpha' = u'\mathbf{x}_u,$$

$$\alpha'' = u''\mathbf{x}_u + (u')^2\mathbf{x}_{uu}.$$

Here u' and u'' are evaluated at s, and $\mathbf{x}_u$ and $\mathbf{x}_{uu}$ are evaluated at $(u(s), v_0)$. Note that $u' = E^{-1/2}$, since the curve has unit speed. Then $T = E^{-1/2}\mathbf{x}_u$ and $N = (EG)^{-1/2}\mathbf{x}_u \wedge \mathbf{x}_v = E^{-1}\mathbf{x}_u \wedge \mathbf{x}_v$. Since $F = 0$ the coordinate system $\mathbf{x}$ is orthogonal, so $V = N \wedge T = E^{-1/2}\mathbf{x}_v$. The geodesic curvature of α is

$$k_g = \langle \alpha'', V \rangle = \frac{(u')^2}{\sqrt{E}} \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = -\frac{E_v}{2E^{3/2}}. \tag{2}$$

Here we used that $\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_u - \langle \mathbf{x}_u, \mathbf{x}_{uv} \rangle = 0 - \frac{1}{2}\langle \mathbf{x}_u, \mathbf{x}_u \rangle_v = -\frac{1}{2} E_v$.

Remark. You may also use (the technique in the proof of) Liouville's Theorem in Chapter 4-4.

4. Since $k_g = k_g(s)$ in (2) is constant, differentiation with respect to s yields

$$0 = k_g' = -\frac{2u'E_{uv}E^{3/2} - 3u'E_uE_vE^{1/2}}{4E^3} = -\frac{2E_{uv}E - 3E_uE_v}{4E^3}.$$

This implies that

$$2E_{uv}E - 3E_uE_v = 0 \tag{3}$$

at every point of U. To see this, let $(u_0, v_0) \in U$, and consider the unit-speed curve $\alpha(s) = \mathbf{x}(u(s), v_0)$ with $u(0) = u_0$. The argument above shows that (3) holds at all points $(u(s), v_0)$, so in particular at $(u(0), v_0) = (u_0, v_0)$.
Next consider the unit-speed curve $\tilde{\alpha}(s) = \mathbf{x}(u_0, v(s))$. Its geodesic curvature is

$$\tilde{k}_g = -\frac{E_u}{2E^{3/2}},$$

where E_u and E are evaluated at $(u_0, v(s))$. A similar derivation as in Part 4 shows that

$$\tilde{k}_g' = -\frac{2E_{uv}E - 3E_uE_v}{4E^3}.$$

Since (3) holds at all points of U, we see that $\tilde{k}_g$ is constant. In other words, the curve $v \mapsto \mathbf{x}(u_0, v)$ has constant geodesic curvature.